

# Entanglement witnesses for $d \otimes d$ systems and new classes of entangled qudit states

D. Chruściński and A. Rutkowski

Institute of Physics, Nicolaus Copernicus University  
Grudziadzka 5, 87–100 Toruń, Poland

## Abstract

We provide a new class of entanglement witnesses for  $d \otimes d$  systems (two qudits). Our construction generalizes the one proposed recently by Jafarizadeh et al. for  $d = 3$  and  $d = 4$  on the basis of semidefinite linear programming. Moreover, we provide a new class of PPT entangled states detected by our witnesses which generalizes well known family of states constructed by Horodecki et al. for  $d = 3$ .

## 1 Introduction

In recent years, due to the rapid development of quantum information theory [1] the necessity of classifying entangled states as a physical resource is of primary importance. It is well known that it is extremely hard to check whether a given density matrix describing a quantum state of the composite system is separable or entangled. There are several operational criteria which enable one to detect quantum entanglement (see e.g. [2] for the recent review). The most famous Peres-Horodecki criterion is based on the partial transposition: if a state  $\rho$  is separable then its partial transposition  $\rho^\Gamma = (\mathbb{1} \otimes T)\rho$  is positive. States which are positive under partial transposition are called PPT states. Clearly each separable state is necessarily PPT but the converse is not true. We stress that it is easy to test whether a given state is PPT, however, there is no general methods to construct PPT states.

The most general approach to characterize quantum entanglement uses a notion of an entanglement witness (EW) [3, 4, 5]. A Hermitian operator  $W$  defined on a tensor product  $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$  is called an EW iff 1)  $\text{Tr}(W\sigma_{\text{sep}}) \geq 0$  for all separable states  $\sigma_{\text{sep}}$ , and 2) there exists an entangled state  $\rho$  such that  $\text{Tr}(W\rho) < 0$  (one says that  $\rho$  is detected by  $W$ ). It turns out that a state is entangled if and only if it is detected by some EW [3]. There was a considerable effort in constructing and analyzing the structure of EWs [6]–[21]. However, there is no general method to construct such objects.

The simplest way to construct EW is to define  $W = P + Q^\Gamma$ , where  $P$  and  $Q$  are positive operators. It is easy to see that  $\text{Tr}(W\sigma_{\text{sep}}) \geq 0$  for all separable states  $\sigma_{\text{sep}}$ , and hence if  $W$  is non-positive, then it is EW. Such EWs are said to be decomposable [6]. Note, however, that decomposable EW cannot detect PPT entangled state and, therefore, such EWs are useless in the search for bound entangled state. Unfortunately, there is no general method to construct non-decomposable EW (nd-EW) and only very few examples of nd-EWs are available in the literature.

In the present paper we provide a class of nd-EWs for  $d \otimes d$  systems. Our construction generalizes the one proposed recently by Jafarizadeh et al. [21] for  $d = 3$  and  $d = 4$  on the basis of semidefinite linear programming. Moreover, we provide a new class of PPT entangled states detected by our witnesses which generalizes well known family of states constructed by Horodecki et al. for  $d = 3$  [22].

The paper is organized as follows: in the next section we introduce a class of circulant operators [23] (see also [24] for more abstract discussion). It turns out that circulant operators defines a natural arena for constructing interesting classes of bi-partite quantum states and the corresponding EWs. Section 3 provides the basic construction of EWs. Then in section 4 we show that our witnesses are non-decomposable by providing a family of PPT entangled states detected by our witnesses. A brief discussion is included in the last section.

## 2 Circulant operators for two qudits

Consider a class of linear Hermitian operators in  $\mathbb{C}^d \otimes \mathbb{C}^d$  constructed as follows: let  $\{|0\rangle, \dots, |d-1\rangle\}$  denotes an orthonormal basis in  $\mathbb{C}^d$  and let  $S : \mathbb{C}^d \rightarrow \mathbb{C}^d$  be a shift operator defined as follows

$$S|k\rangle = |k+1\rangle, \quad (\text{mod } d). \quad (1)$$

One introduces the following family of  $d$ -dimensional subspaces in  $\mathbb{C}^d \otimes \mathbb{C}^d$ :

$$\Sigma_0 = \text{span}\{|00\rangle, \dots, |d-1, d-1\rangle\}, \quad (2)$$

and

$$\Sigma_n = (\mathbb{I} \otimes S^n) \Sigma_0, \quad n = 1, \dots, d-1. \quad (3)$$

It is clear that  $\Sigma_m$  and  $\Sigma_n$  are mutually orthogonal for  $m \neq n$  and hence the collection  $\{\Sigma_0, \dots, \Sigma_{d-1}\}$  defines direct sum decomposition of  $\mathbb{C}^d \otimes \mathbb{C}^d$

$$\mathbb{C}^d \otimes \mathbb{C}^d = \Sigma_0 \oplus \dots \oplus \Sigma_{d-1}. \quad (4)$$

Following [23] we call (4) a *circulant decomposition*. Now we construct a circulant Hermitian operator corresponding to the circulant decomposition (4). Let us introduce a set of Hermitian  $d \times d$  matrices  $a^{(n)} = [a_{ij}^{(n)}]$ ;  $n = 0, 1, \dots, d-1$ , and define Hermitian operators  $A_n$  supported on  $\Sigma_n$  via the following formula

$$\begin{aligned} A_n &= \sum_{i,j=0}^{d-1} a_{ij}^{(n)} |i\rangle\langle j| \otimes S^n |i\rangle\langle j| S^{\dagger n} \\ &= \sum_{i,j=0}^{d-1} a_{ij}^{(n)} |i\rangle\langle j| \otimes |i+n\rangle\langle j+n|. \end{aligned} \quad (5)$$

Finally, we define the circulant Hermitian operator

$$A = A_0 + A_1 + \dots + A_{d-1}. \quad (6)$$

Note, that if all  $A_n$  are semipositive definite and  $\text{Tr } A = 1$ , then  $A$  defines a legitimate quantum state of two qudits called *circulant state* [23]. Interestingly, many well known examples of quantum states of composite  $d \otimes d$  systems belong to this class (see [23, 24] for examples).

The crucial property of circulant operators is based on the following observation: the partially transposed circulant operator  $A$  displays similar circulant structure, that is,

$$A^\Gamma = \tilde{A}_0 \oplus \dots \oplus \tilde{A}_{d-1}, \quad (7)$$

where the Hermitian operators  $\tilde{A}_n$  are supported on the new collection of subspaces  $\tilde{\Sigma}_n$  which are defined as follows:

$$\tilde{\Sigma}_0 = \text{span}\{|0, \pi(0)\rangle, |1, \pi(1)\rangle, \dots, |d-1, \pi(d-1)\rangle\}, \quad (8)$$

where  $\pi$  is a permutation defined by

$$\pi(k) = d - k, \quad (\text{mod } d). \quad (9)$$

The remaining subspaces  $\tilde{\Sigma}_n$  are defined by a cyclic shift

$$\tilde{\Sigma}_n = (\mathbb{I} \otimes S^n) \tilde{\Sigma}_0, \quad n = 1, \dots, d-1. \quad (10)$$

Again, the collection  $\{\tilde{\Sigma}_0, \dots, \tilde{\Sigma}_{d-1}\}$  defines direct sum decomposition of  $\mathbb{C}^d \otimes \mathbb{C}^d$

$$\mathbb{C}^d \otimes \mathbb{C}^d = \tilde{\Sigma}_0 \oplus \dots \oplus \tilde{\Sigma}_{d-1}. \quad (11)$$

Moreover, operators  $\tilde{A}_n$  are defined as follows

$$\begin{aligned} \tilde{A}_n &= \sum_{i,j=0}^{d-1} \tilde{a}_{ij}^{(n)} |i\rangle\langle j| \otimes S^n |\pi(i)\rangle\langle\pi(j)| S^{n\dagger} \\ &= \sum_{i,j=0}^{d-1} \tilde{a}_{ij}^{(n)} |i\rangle\langle j| \otimes |\pi(i) + n\rangle\langle\pi(j) + n|, \end{aligned} \quad (12)$$

with

$$\tilde{a}^{(n)} = \sum_{m=0}^{d-1} a^{(n+m)} \circ (\Pi S^m), \quad (\text{mod } d), \quad (13)$$

where  $\Pi$  is a permutation matrix corresponding to  $\pi$ , that is

$$\Pi_{kl} = \delta_{k, \pi(l)}, \quad (14)$$

and  $a \circ b$  denotes the Hadamard product of  $d \times d$  matrices  $a$  and  $b$ , that is,  $(a \circ b)_{ij} = a_{ij} b_{ij}$  [25].

### 3 Entanglement witnesses for $d \otimes d$ systems

Following [21] one constructs the following circulant operators in  $\mathbb{C}^d \otimes \mathbb{C}^d$

$$O_0 = \frac{1}{d} \sum_{i=0}^{d-1} |ii\rangle\langle ii| \quad (15)$$

and

$$O_n = (\mathbb{I} \otimes S^n) O_0 (\mathbb{I} \otimes S^n)^\dagger = \frac{1}{d} \sum_{i=0}^{d-1} |i, i+n\rangle\langle i, i+n|, \quad (16)$$

for  $n = 1, \dots, d-1$ . Note that  $O_\alpha$  defines a normalized, i.e.  $\text{Tr} O_\alpha = 1$ , projector onto  $\Sigma_\alpha$ . Finally, let  $P_d^+$  denotes the projector onto the maximally entangled states in  $\mathbb{C}^d \otimes \mathbb{C}^d$ , that is

$$P_d^+ = \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle\langle jj|. \quad (17)$$

Note that  $O_m O_n = \frac{1}{d} \delta_{mn} O_m$ , and  $O_m P_d^+ = 0$  for  $n \geq 1$ . Moreover,

$$O_0 + O_1 + \dots + O_{d-1} = \frac{1}{d} \mathbb{I}_d \otimes \mathbb{I}_d. \quad (18)$$

Let us consider the following family (parametrized by  $\alpha$ ) of circulant Hermitian operators

$$W_\alpha = \mathbb{I}_d \otimes \mathbb{I}_d - \frac{1}{\alpha} O_1 - d(O_2 + \dots + O_{d-1}) - \left(2 - \frac{1}{d\alpha}\right) P_d^+, \quad (19)$$

together with

$$W'_\alpha = \mathcal{P} W_\alpha \mathcal{P}, \quad (20)$$

where  $\mathcal{P}$  denotes permutation (flip/swap) operator defined by

$$\mathcal{P} = \sum_{i,j=0}^{d-1} |ij\rangle\langle ji| . \quad (21)$$

One easily finds

$$\mathcal{P}O_k\mathcal{P} = O_{\pi(k)} , \quad (22)$$

with  $\pi$  defined in (9). Moreover,  $\mathcal{P}P_d^+\mathcal{P} = P_d^+$ . Hence

$$W'_\alpha = \mathbb{I}_d \otimes \mathbb{I}_d - \frac{1}{\alpha}O_{d-1} - d(O_1 + O_2 + \dots O_{d-2}) - \left(2 - \frac{1}{d\alpha}\right)P_d^+ . \quad (23)$$

Note that for  $d = 3$  and  $d = 4$  one recovers Eqs. (17) and (38), respectively, from Jafarizadeh et al. [21]. Note that using (18) one obtains a simplified formulae

$$W_\alpha = dO_0 + \left(d - \frac{1}{\alpha}\right)O_1 - \left(2 - \frac{1}{d\alpha}\right)P_d^+ , \quad (24)$$

and

$$W'_\alpha = dO_0 + \left(d - \frac{1}{\alpha}\right)O_{d-1} - \left(2 - \frac{1}{d\alpha}\right)P_d^+ . \quad (25)$$

Hence, up to a factor  $\mu = 2 - \frac{1}{d\alpha}$ ,  $W_\alpha$  and  $W'_\alpha$  belong to a class

$$\mathbf{W}[a_0, a_1, \dots, a_{d-1}] = (a_0 + 1)O_0 + \sum_{n=1}^{d-1} a_n O_n - P_d^+ , \quad (26)$$

that is,

$$W_\alpha = \mu \mathbf{W}[a_0, a_1, \dots, a_{d-1}] , \quad (27)$$

with

$$\begin{aligned} a_0 &= \frac{d}{\mu} - 1 \\ a_1 &= \frac{d}{\mu} - \frac{1}{\alpha\mu} , \\ a_2 &= a_3 = \dots = a_{d-1} = 0 . \end{aligned} \quad (28)$$

Clearly, one obtains  $W'_\alpha$  by interchanging  $a_1$  and  $a_{d-1}$ . Unfortunately, we do not know necessary and sufficient conditions for  $\mathbf{W}[a_0, a_1, \dots, a_{d-1}]$  to be entanglement witness. Clearly,  $a_0, \dots, a_{d-1} \geq 0$ . Moreover, one easily shows that necessarily

$$a_0 + a_1 + \dots + a_{d-1} \geq d - 1 . \quad (29)$$

Indeed, taking  $\psi = \sum_{i=0}^{d-1} |i\rangle \in \mathbb{C}^d$ , one recovers (29) from  $\langle \psi \otimes \psi | \mathbf{W}[a_0, a_1, \dots, a_{d-1}] | \psi \otimes \psi \rangle \geq 0$ .

Finally,  $\mathbf{W}[a_0, a_1, \dots, a_{d-1}]$  becomes a positive operator if and only if  $a_0 \geq d - 1$ . Hence, necessarily

$$a_0 < d - 1 . \quad (30)$$

The problem is completely solved only for  $d = 3$  [27]. In this case apart from (29) and (30) one has an additional condition which says that if  $a_0 \leq 1$ , then

$$a_1 a_2 \geq (1 - a_0)^2 . \quad (31)$$

Moreover, for  $d = 3$  we know that an entanglement witness  $\mathbf{W}[a_0, a_1, a_2]$  is non-decomposable if and only if [27]

$$a_1 a_2 < \frac{(2 - a_0)^2}{4} . \quad (32)$$

For  $d > 3$  we know only special cases when  $\mathbf{W}[a_0, a_1, \dots, a_{d-1}]$  defines an EW. In particular it is well known [28] (see also [14, 15]) that  $\mathbf{W}[d-2, 1, 0, \dots, 0]$  defines nd-EW.

Consider now a family of Hermitian operators defined in (27). Note that

$$a_0 + a_1 + \dots + a_{d-1} = a_0 + a_1 = d - 1 , \quad (33)$$

and hence the necessary condition (29) to be an EW is satisfied. Now, to satisfy (30) one finds  $\alpha > \frac{1}{d}$ . Let us observe that a convex combination

$$\begin{aligned} & a_1 \mathbf{W}[d-2, 1, 0, \dots, 0] + (1 - a_1) \mathbf{W}[d-1, 0, 0, \dots, 0] \\ &= \mathbf{W}[(d-1) - a_1, a_1, 0, \dots, 0] , \end{aligned} \quad (34)$$

defines an EW for any  $a_1 \in (0, 1]$ . Now, the condition  $a_1 \leq 1$  implies  $\alpha \leq \frac{d-1}{d(d-2)}$ . Hence, we have shown that for any

$$\frac{1}{d} < \alpha \leq \frac{d-1}{d(d-2)} , \quad (35)$$

the corresponding operator  $W_\alpha$  defines an EW.

Let us observe that for  $d = 3$  formula (35) reproduces analytical result  $\alpha \in (\frac{1}{3}, \frac{2}{3}]$  from Jafarizadeh et al. [21]. Note, however, that for  $d = 4$  the numerical result  $\alpha \in (\frac{1}{4}, \frac{1}{3}]$  from [21] is improved. Formula (35) gives for  $d = 4$  the following analytical condition  $\alpha \in (\frac{1}{4}, \frac{3}{8}] \supset (\frac{1}{4}, \frac{1}{3}]$ .

## 4 A family of 2-qudit states

In this section we show that whenever  $\alpha$  satisfies (35) then  $W_\alpha$  and  $W'_\alpha$  are non-decomposable EW. In order to show it one has to construct a PPT entangled state  $\rho$  such that  $\text{Tr}(W_\alpha \rho) < 0$ . Consider the following family of circulant 2-qudit states

$$\rho = \sum_{i=1}^{d-1} \lambda_i O_i + \lambda_d P_d^+ , \quad (36)$$

with  $\lambda_n \geq 0$ , and  $\lambda_1 + \dots + \lambda_{d-1} + \lambda_d = 1$ . One easily finds

$$\text{Tr}(W_\alpha \rho) = (\lambda_1 - \lambda_d) \left( 1 - \frac{1}{d\alpha} \right) , \quad (37)$$

and

$$\text{Tr}(W'_\alpha \rho) = (\lambda_{d-1} - \lambda_d) \left( 1 - \frac{1}{d\alpha} \right) , \quad (38)$$

Let us take the following special case corresponding to

$$\begin{aligned} \lambda_1 &= \frac{\beta}{\ell} , \\ \lambda_{d-1} &= \frac{(d-1)^2 + 1 - \beta}{\ell} , \\ \lambda_d &= \frac{d-1}{\ell} . \end{aligned} \quad (39)$$

and  $\lambda_2 = \dots = \lambda_{d-2} = \lambda_d$ , with

$$\ell = (d-1)(2d-3) + 1 . \quad (40)$$

The parameter  $\beta \in [0, (d-1)^2 + 1]$ . One has

$$\text{Tr}(W_\alpha \rho) = \frac{\beta - (d-1)}{\ell} \left( 1 - \frac{1}{d\alpha} \right) , \quad (41)$$

and

$$\text{Tr}(W'_\alpha \rho) = \frac{(d-1)(d-2) + 1 - \beta}{\ell} \left(1 - \frac{1}{d\alpha}\right), \quad (42)$$

It is easy to see that a state  $\rho$  defined by (39) is PPT iff  $\lambda_1 \lambda_{d-1} \geq \lambda_d^2$  [26]. It gives the following condition for the parameter  $\beta$

$$1 \leq \beta \leq (d-1)^2. \quad (43)$$

Note, that for  $d = 3$  formula (36) with  $\lambda$ s defined in (39)

$$\rho = \frac{2}{7}P_3^+ + \frac{\beta}{7}O_1 + \frac{5-\beta}{7}O_2, \quad (44)$$

reproduces well know family of Horodecki states [22], and (43) reproduces well known PPT condition:  $1 \leq \beta \leq 4$ . Actually, in this case a state is separable for  $\beta \in [2, 3]$ . It is PPT entangled for  $\beta \in [1, 2) \cup (3, 4]$ , and NPT for  $\beta \in [0, 1) \cup (4, 5]$ . For  $d = 4$  our family reproduces a state considered in [21] (see Eq. (43) with  $\gamma = 3$ ).

It is clear that for  $\beta < d-1$  and  $\alpha$  satisfying (35)  $W_\alpha$  detects entanglement of  $\rho$  which proves that  $W_\alpha$  is non-decomposable EW. Similarly, for  $\beta > (d-1)(d-2)+1$  and  $\alpha$  satisfying (35)  $W'_\alpha$  detects entanglement of  $\rho$  which proves that  $W'_\alpha$  is non-decomposable EW. As a byproduct we showed that for

$$\beta \in [1, d-1) \cup ((d-1)(d-2)+1, (d-1)^2+1]$$

a state  $\rho$  is PPT entangled.

## 5 Conclusions

We provide a new class of non-decomposable entanglement witnesses for  $d \otimes d$  systems (two qudits). Our construction generalizes the one proposed recently by Jafarizadeh et al. [21] for  $d = 3$  and  $d = 4$  on the basis of semidefinite linear programming. We stress that for  $d = 3$  we recover analytical result of [21]. However, for  $d = 4$  our analytical result slightly improves numerical result of [21]. As a byproduct, we provided a new class of PPT entangled states detected by our witnesses which generalizes well known family of states constructed by Horodecki et al. for  $d = 3$  [22].

For the experimental realization of entanglement witnesses  $W_\alpha$  and  $W'_\alpha$  discussed in this paper one can use for example the generalized Gell-Mann matrix basis (see e.g. [29, 30]) consisting in the following set of Hermitian matrices: symmetric

$$\Lambda_s^{k\ell} = |k\rangle\langle\ell| + |\ell\rangle\langle k|, \quad 0 \leq k < \ell \leq d-1, \quad (45)$$

antisymmetric

$$\Lambda_a^{k\ell} = -i|k\rangle\langle\ell| + i|\ell\rangle\langle k|, \quad 0 \leq k < \ell \leq d-1, \quad (46)$$

and diagonal

$$\Lambda^\ell = \sqrt{\frac{2}{\ell(\ell+1)}} \left( \sum_{j=0}^{\ell-1} |j\rangle\langle j| - (\ell-1)|\ell\rangle\langle\ell| \right), \quad (47)$$

for  $1 \leq \ell \leq d-2$ . For  $d = 3$  the above set reconstructs the standard  $3 \times 3$  Gell-Mann matrices. Let us observe that in this case Gell-Mann matrices may be defined in terms of spin-1 operators

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad (48)$$

and

$$S_z = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (49)$$

One finds the following relations [30]

$$\begin{aligned} \Lambda_s^{01} &= \frac{1}{\sqrt{2}\hbar^2} (\hbar S_x + \{S_z, S_x\}) \\ \Lambda_s^{02} &= \frac{1}{\hbar^2} (S_x^2 - S_y^2), \\ \Lambda_s^{12} &= \frac{1}{\sqrt{2}\hbar^2} (\hbar S_x - \{S_z, S_x\}), \\ \Lambda_a^{01} &= \frac{1}{\sqrt{2}\hbar^2} (\hbar S_y + \{S_y, S_z\}), \\ \Lambda_a^{02} &= \frac{1}{\hbar^2} \{S_x, S_y\} \\ \Lambda_a^{12} &= \frac{1}{\sqrt{2}\hbar^2} (\hbar S_y - \{S_y, S_z\}), \\ \Lambda^0 &= 2\mathbb{I}_3 + \frac{1}{\sqrt{2}\hbar^2} (\hbar S_z - 3S_x^2 - 3S_y^2), \\ \Lambda^1 &= \frac{1}{\sqrt{3}} \left( -2\mathbb{I}_3 + \frac{3}{2\hbar^2} (\hbar S_z + S_x^2 + S_y^2) \right), \end{aligned}$$

where  $\{A, B\} = AB + BA$ . Finally, one finds the following representation of  $W_\alpha$  in terms of local Hermitian operators

$$\begin{aligned} W_\alpha &= \left(4 - \frac{2}{3\alpha}\right) \mathbb{I}_3 \otimes \mathbb{I}_3 - \left(\frac{1}{3} - \frac{5}{9\alpha}\right) [\Lambda^0 \otimes \Lambda^0 \\ &+ \Lambda^1 \otimes \Lambda^1] + \sqrt{3} \left(1 - \frac{1}{3\alpha}\right) [\Lambda^0 \otimes \Lambda^1 - \Lambda^1 \otimes \Lambda^0] \\ &- \left(\frac{4}{3} - \frac{2}{9\alpha}\right) [\Lambda_s^{01} \otimes \Lambda_s^{01} + \Lambda_s^{02} \otimes \Lambda_s^{02} + \Lambda_s^{12} \otimes \Lambda_s^{12} \\ &- \Lambda_a^{01} \otimes \Lambda_a^{01} - \Lambda_a^{02} \otimes \Lambda_a^{02} - \Lambda_a^{12} \otimes \Lambda_a^{12}]. \end{aligned} \quad (50)$$

Similar decomposition can be found for  $W'_\alpha$ . Hence, to measure entanglement witnesses  $W_\alpha$  and  $W'_\alpha$  one can perform suitable number of purely local measurement settings  $\Lambda^\mu \otimes \Lambda^\nu$ . Using the general scheme (45)-(47) one can represent  $W_\alpha$  and  $W'_\alpha$  for arbitrary (finite)  $d$  as a combination of purely local observables. In this way one can experimentally find out whether a given 2-qudit state is entangled or not.

## Acknowledgments

This work was partially supported by the Polish Ministry of Science and Higher Education Grant No 3004/B/H03/2007/33.

## References

- [1] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge, 2000.
- [2] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

- [3] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
- [4] B. Terhal, Phys. Lett. A **271**, 319 (2000); Linear Algebr. Appl. **323**, 61 (2000).
- [5] B. M. Terhal, Theor. Comput. Sci. **287**, 313 (2002).
- [6] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A **62**, 052310 (2000).
- [7] M. Lewenstein, B. Kraus, P. Horodecki, and J. I. Cirac, Phys. Rev. A **63**, 044304 (2001).
- [8] B. Kraus, M. Lewenstein, and J. I. Cirac, Phys. Rev. A **65**, 042327 (2002).
- [9] P. Hyllus, O. Gühne, D. Bruß, and M. Lewenstein Phys. Rev. A **72**, 012321 (2005).
- [10] D. Bruß, J. Math. Phys. **43**, 4237 (2002).
- [11] G. Tóth and O. Gühne, Phys. Rev. Lett. **94**, 060501 (2005)
- [12] R. A. Bertlmann, H. Narnhofer and W. Thirring, Phys. Rev. A **66**, 032319 (2002).
- [13] H.-P. Breuer, Phys. Rev. Lett. **97**, 080501 (2006).
- [14] D. Chruściński and A. Kossakowski, Open Systems and Inf. Dynamics, **14**, 275-294 (2007).
- [15] D. Chruściński and A. Kossakowski, J. Phys. A: Math. Theor. **41**, 145301 (2008); J. Phys. A: Math. Theor. **41**, 215201 (2008).
- [16] D. Chruściński and A. Kossakowski, Comm. Math. Phys. **290**, 1051 (2009).
- [17] D. Chruściński, A. Kossakowski and G. Sarbicki, Phys. Rev. A **80**, 042314 (2009).
- [18] D. Chruściński, J. Pytel and G. Sarbicki, Phys. Rev. A **80**, 062314 (2009).
- [19] D. Chruściński and J. Pytel, Phys. Rev. A **82**, 052310 (2010).
- [20] M.A. Jafarizadeh and N. Behzadi, Eur. Phys. J. D **55**, 729 (2009).
- [21] M.A. Jafarizadeh, N. Behzadi and Y. Akbari, Eur. Phys. J. D **55**, 197 (2009).
- [22] P. Horodecki, M. Horodecki, and R. Horodecki, Phys. Rev. Lett. **82**, 1056 (1999).
- [23] D. Chruściński and A. Kossakowski, Phys. Rev. A **76**, 032308 (2007).
- [24] D. Chruściński and A. Pittenger, J. Phys. A: Math. Theor. **41**, 385301 (2008).
- [25] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, (Cambridge University Press, New York, 1991).
- [26] D. Chruściński and A Kossakowski, Phys. Rev. A **74**, 022308 (2006).
- [27] S. J. Cho, S.-H. Kye, and S. G. Lee, Linear Algebr. Appl. **171**, 213 (1992).
- [28] H. Osaka, Linear Algebr. Appl. **186**, 45 (1993).
- [29] G. Kimura, Phys. Lett. **314**, 339 (2003).
- [30] R. A. Bertlmann and P. Krammer, J. Phys. A: Math. Theor. **41**, 235303 (2008).